MODULE – II

HOOKE’S LAW

PLANE PROBLEMS N ELASTICITY

GENERALIZED HOOKE’S LAW (STRESS – STRAIN)

σ	ε	Young’s Modulus
τ	γ	Modulus of Rigidity
Pressure	Volumetric strain	Bulk Modulus

GENERALIZED HOOKE'S LAW (STRESS – STRAIN)

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_{zz} \end{bmatrix}$$

Constitutive relations or Generalized Hooke's law relates the state of stress at a point to the state of strain at the same point. It describes the behavior of a material not the behavior of a body.

GENERALIZED HOOKE'S LAW (STRESS – STRAIN)

The nine rectangular components of stress are related to the nine rectangular components of strains and there will be **81** elastic constants.

Due to the equality of cross shear there are six independent components of stress and six independent components of strain, 81 elastic constants reduces to **36**.

GENERALIZED HOOKE'S LAW (STRESS – STRAIN)

In most generalized case mathematically expression for the above statement can be written as :

$$\sigma_{xx} = a_{11}\epsilon_{xx} + a_{12}\epsilon_{yy} + a_{13}\epsilon_{zz} + a_{14}\gamma_{xy} + a_{15}\gamma_{yz} + a_{16}\gamma_{xz}$$

$$\sigma_{yy} = a_{21}\epsilon_{xx} + a_{22}\epsilon_{yy} + a_{23}\epsilon_{zz} + a_{24}\gamma_{xy} + a_{25}\gamma_{yz} + a_{26}\gamma_{xz}$$

$$\sigma_{zz} = a_{31}\epsilon_{xx} + a_{32}\epsilon_{yy} + a_{33}\epsilon_{zz} + a_{34}\gamma_{xy} + a_{35}\gamma_{yz} + a_{36}\gamma_{xz}$$

$$\tau_{xy} = a_{41}\epsilon_{xx} + a_{42}\epsilon_{yy} + a_{43}\epsilon_{zz} + a_{44}\gamma_{xy} + a_{45}\gamma_{yz} + a_{46}\gamma_{xz}$$

$$\tau_{yz} = a_{51}\epsilon_{xx} + a_{52}\epsilon_{yy} + a_{53}\epsilon_{zz} + a_{54}\gamma_{xy} + a_{55}\gamma_{yz} + a_{56}\gamma_{xz}$$

$$\tau_{xz} = a_{61}\epsilon_{xx} + a_{62}\epsilon_{yy} + a_{63}\epsilon_{zz} + a_{64}\gamma_{xy} + a_{65}\gamma_{yz} + a_{66}\gamma_{xz}$$

GENERALIZED HOOKE'S LAW (STRESS – STRAIN)

Conversely the strain – stress can written as:

$$\epsilon_{xx} = b_{11}\sigma_{xx} + b_{12}\sigma_{yy} + b_{13}\sigma_{zz} + b_{14}\tau_{xy} + b_{15}\tau_{yz} + b_{16}\tau_{xz}$$

$$\epsilon_{yy} = b_{21}\sigma_{xx} + b_{22}\sigma_{yy} + b_{23}\sigma_{zz} + b_{24}\tau_{xy} + b_{25}\tau_{yz} + b_{26}\tau_{xz}$$

$$\epsilon_{zz} = b_{31}\sigma_{xx} + b_{32}\sigma_{yy} + b_{33}\sigma_{zz} + b_{34}\tau_{xy} + b_{35}\tau_{yz} + b_{36}\tau_{xz}$$

$$\gamma_{xy} = b_{41}\sigma_{xx} + b_{42}\sigma_{yy} + b_{43}\sigma_{zz} + b_{44}\tau_{xy} + b_{45}\tau_{yz} + b_{46}\tau_{xz}$$

$$\gamma_{yz} = b_{51}\sigma_{xx} + b_{52}\sigma_{yy} + b_{53}\sigma_{zz} + b_{54}\tau_{xy} + b_{55}\tau_{yz} + b_{56}\tau_{xz}$$

$$\gamma_{xz} = b_{61}\sigma_{xx} + b_{62}\sigma_{yy} + b_{63}\sigma_{zz} + b_{64}\tau_{xy} + b_{65}\tau_{yz} + b_{66}\tau_{xz}$$

GENERALIZED HOOKE'S LAW (STRESS – STRAIN)

For a homogeneous linearly elastic, non –isotropic material, the first and second set of equations are known as **generalized Hooke's law.**

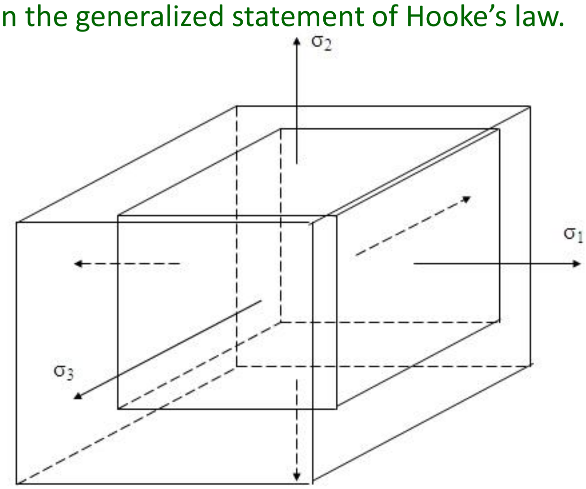
If the material property of a material is independent of material position, such a material is called **homogeneous material.**

If the material property of the material are independent of direction, such a material is called **isotropic material.**

A material whose properties are dependent on direction is called **anisotropic material.** Eg: wood, fiber reinforced composite material.

HOOKE'S LAW FOR ISOTROPIC MATERIALS

For an isotropic material there are only two independent elastic constants in the generalized statement of Hooke's law.



HOOKE'S LAW FOR ISOTROPIC MATERIALS

The relation between the 3 principal stresses and 3 principal strains can written as :

$$\sigma_1 = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3$$

Where a, b and c are constants.

The effect of σ_1 along the directions 2 & 3 are same. So the above equation can be written as :

$$\sigma_1 = a\varepsilon_1 + b(\varepsilon_2 + \varepsilon_3)$$

By adding and subtracting $b\varepsilon_1$,

$$\sigma_1 = (a - b)\varepsilon_1 + b(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Put a-b = 2μ and b = λ

$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \Delta$ – Cubical Dilatation.

$$\left.\begin{aligned} \sigma_1 &= \lambda\Delta + 2\mu\varepsilon_1 \\ \sigma_2 &= \lambda\Delta + 2\mu\varepsilon_2 \\ \sigma_3 &= \lambda\Delta + 2\mu\varepsilon_3 \end{aligned}\right\} \quad (1)$$

Here λ and μ are called **<u>Lami's constant</u>**

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Modulus of Rigidity:

Let the co ordinate axis at a point P coincide with the principal stress axis i.e., the co ordinate axis are along 1, 2 & 3.

For an isotropic material, the principal axis of strain will also coincide with ox, oy and oz.

Consider another frame of reference ox', oy' and oz' whose direction cosines are given below:

$$\begin{matrix} n_{xx'} & n_{xy'} & n_{xz'} \\ n_{yx'} & n_{yy'} & n_{yz'} \\ n_{zx'} & n_{zy'} & n_{zz'} \end{matrix}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$n_{xx'} \cdot n_{xy'} + n_{yx'} \cdot n_{yy'} + n_{zx'} \cdot n_{zy'} = 0 \text{-------------} \quad (2)$$

The normal stresses, $\sigma_{x'x'}$ and $\tau_{x'y'}$ are obtained using the stress transformation eqn.

$$\left.\begin{aligned} \sigma_{x'x'} &= \sigma_1 n_{xx'}^2 + \sigma_2 n_{yx'}^2 + \sigma_3 n_{zx'}^2 \\ \tau_{x'y'} &= \sigma_1 n_{xx'} n_{xy'} + \sigma_2 n_{yx'} n_{yy'} + \sigma_3 n_{zx'} n_{zy'} \end{aligned}\right\} \quad \textbf{(3)}$$

Similarly $\boldsymbol{\varepsilon_{x'x'}}$ **& ½** $\boldsymbol{\gamma_{x'y'}}$ can be written as

$$\left.\begin{aligned} \varepsilon_{x'x'} &= \varepsilon_1 {n_{xx'}}^2 + \varepsilon_2 {n_{yx'}}^2 + \varepsilon_3 {n_{zx'}}^2 \\ \frac{1}{2}\gamma_{x'y'} &= \varepsilon_1 n_{xx'} n_{xy'} + \varepsilon_2 n_{yx'} n_{yy'} + \varepsilon_3 n_{zx'} n_{zy'} \end{aligned}\right\} \quad \textbf{(4)}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Substituting for σ_1, σ_2, σ_3 from equ (1) in eq. 3 we get

$$\tau_{x'y'} = \lambda\Delta\left(n_{xx'}.n_{xy'} + n_{yx'}.n_{yy'} + n_{zx'}.n_{zy'}\right) + 2\mu\left(\varepsilon_1 n_{xx'}.n_{xy'} + \varepsilon_2 n_{yx'}.n_{yy'} + \varepsilon_3\, n_{zx'}.n_{zy'}\right)$$

Using eqn 2 and eqn 4 the above becomes

$$\tau_{x'y'} = \mu\gamma_{x'y'} \qquad \text{---------} \quad (5)$$

Modulus of rigidity G is defined as the ratio of shear stress to shear strain.

Thus, $\mathbf{G = \mu}$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Bulk Modulus:

Let A & B be two points with co ordinates (x, y, z) and (x+Δx, y+Δy z+Δz) before deformation.

After deformation point A moved by u, v, w along x, y, z direction, point B moves by distance

$$u + \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$v + \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \frac{\partial v}{\partial z}\Delta z$$

$$w + \frac{\partial w}{\partial x}\Delta x + \frac{\partial w}{\partial y}\Delta y + \frac{\partial w}{\partial z}\Delta z$$

Along x, y, z direction respectively.

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Components of AB are Δx, Δy, & Δz.

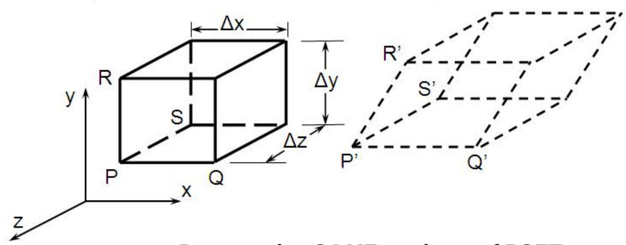
Components of A'B' are :

$$\left(1+\frac{\partial u}{\partial x}\right)\Delta x+\frac{\partial u}{\partial y}\Delta y+\frac{\partial u}{\partial z}\Delta z$$

$$\frac{\partial v}{\partial x}\Delta x+\left(1+\frac{\partial v}{\partial y}\right)\Delta y+\frac{\partial v}{\partial z}\Delta z$$

$$\frac{\partial w}{\partial x}\Delta x+\frac{\partial w}{\partial y}\Delta y+\left(1+\frac{\partial w}{\partial z}\right)\Delta z$$

Consider a parallelepiped as shown in figure below



HOOKE'S LAW FOR ISOTROPIC MATERIALS

PQ parallel to x axis, PR parallel to y axis and PS parallel to z axis.

After the deformation the new shape of the parallelepiped is shown in figure in dotted lines.

Components of PQ, PR and PS are Δx, 0, 0; 0, Δy, 0; 0, 0, Δz respectively.

V be the original volume.

V + ΔV be the final volume.

HOOKE'S LAW FOR ISOTROPIC MATERIALS

The components of P'Q', P'R', P'S' are given below

	P'Q'	P'R'	P'S'
x	$\left(1+\frac{\partial u}{\partial x}\right)\Delta x$	$\frac{\partial u}{\partial y}\Delta y$	$\frac{\partial u}{\partial z}\Delta z$
y	$\frac{\partial v}{\partial x}\Delta x$	$\left(1+\frac{\partial v}{\partial y}\right)\Delta y$	$\frac{\partial v}{\partial z}\Delta z$
z	$\frac{\partial w}{\partial x}\Delta x$	$\frac{\partial w}{\partial y}\Delta y$	$\left(1+\frac{\partial w}{\partial z}\right)\Delta z$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Volume of the parallelepiped after deformation is given by

$$V+\Delta V=\begin{vmatrix}\left(1+\frac{\partial u}{\partial x}\right)\Delta x & \frac{\partial u}{\partial y}\Delta y & \frac{\partial u}{\partial z}\Delta z\\ \frac{\partial v}{\partial x}\Delta x & \left(1+\frac{\partial v}{\partial y}\right)\Delta y & \frac{\partial v}{\partial z}\Delta z\\ \frac{\partial w}{\partial x}\Delta x & \frac{\partial w}{\partial y}\Delta y & \left(1+\frac{\partial w}{\partial z}\right)\Delta z\end{vmatrix}$$

$$V+\Delta V=\begin{vmatrix}\left(1+\frac{\partial u}{\partial x}\right) & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \left(1+\frac{\partial v}{\partial y}\right) & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \left(1+\frac{\partial w}{\partial z}\right)\end{vmatrix}\Delta x\Delta y\Delta z$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Expanding and neglecting the product of derivatives.

$$\mathbf{V + \Delta V = \left(1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \Delta x \Delta y \Delta z}$$

$$\mathbf{V + \Delta V = \left(1 + \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) \Delta x \Delta y \Delta z}$$

Volumetric Strain = $\mathbf{\dfrac{V + \Delta V - V}{V}}$

$$= \mathbf{\frac{\left(1 + \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right) \Delta x \Delta y \Delta z - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}}$$

Volumetric Strain = $\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = \epsilon_1 + \epsilon_2 + \epsilon_3 = \Delta$ --- (6)

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Cubical dilatation is the volumetric strain which is equal to first strain invariant.

Using eq. 3,

$$\boldsymbol{\sigma_{x'x'} = \sigma_1 n_{xx'}^2 + \sigma_2 n_{yx'}^2 + \sigma_3 n_{zx'}^2}$$

$$\sigma_{x'x'} = (\lambda\Delta + 2\mu\varepsilon_1)\, n_{xx'}^2 + (\lambda\Delta + 2\mu\varepsilon_2)\, n_{yx'}^2 + (\lambda\Delta + 2\mu\varepsilon_3)\, n_{zx'}^2$$

$$\sigma_{x'x'} = \lambda\Delta\left(n_{xx'}{}^2 + n_{yx'}{}^2 + n_{zx'}{}^2\right) + 2\mu\left(\varepsilon_1 n_{xx'}{}^2 + \varepsilon_2 n_{yx'}{}^2 + \varepsilon_3 n_{zx'}{}^2\right)$$

(using eq. 4)

HOOKE'S LAW FOR ISOTROPIC MATERIALS

So,

$$\left.\begin{aligned} \sigma_{x'x'} &= \lambda\Delta + 2\mu\varepsilon_{x'x'} \\ \sigma_{y'y'} &= \lambda\Delta + 2\mu\varepsilon_{y'y'} \\ \sigma_{z'z'} &= \lambda\Delta + 2\mu\varepsilon_{z'z'} \end{aligned}\right\} \quad (7)$$

Adding the above 3 eqns. We have

$$\sigma_{x'x'} + \sigma_{y'y'} + \sigma_{z'z'} = (3\lambda + 2\mu)\Delta$$

$$\sigma_1 + \sigma_2 + \sigma_3 = (3\lambda + 2\mu)\Delta \quad \text{----} \quad (8)$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

When $\sigma_1 = \sigma_2 = \sigma_3 = P$, We have $3P = (3\lambda + 2\mu)\Delta$

Bulk Modulus, $K = \dfrac{\text{Pressure}}{\text{Volumetric Strain}}$

$$\text{Bulk Modulus}, K = \frac{P}{\Delta} = \lambda + \frac{2}{3}\mu \qquad (9)$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Young's Modulus & Poisson's Ratio:

From eqn. 8 we have
$$\Delta = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3\lambda + 2\mu}$$

Substituting this in equ. 1

$$\sigma_1 = \frac{\lambda}{3\lambda+2\mu}(\sigma_1 + \sigma_2 + \sigma_3) + 2\mu\varepsilon_1$$

$$2\mu\varepsilon_1 = \frac{3\lambda\sigma_1 + 2\mu\sigma_1 - \lambda\sigma_1 - \lambda(\sigma_2 + \sigma_3)}{3\lambda + 2\mu}$$

$$\varepsilon_1 = \frac{2(\lambda + \mu)\sigma_1 - \lambda(\sigma_2 + \sigma_3)}{2\mu(3\lambda + 2\mu)}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$\varepsilon_1 = \frac{(\lambda + \mu)}{\mu(3\lambda + 2\mu)}\left[\sigma_1 - \frac{\lambda(\sigma_2 + \sigma_3)}{2(\lambda + \mu)}\right] \quad (10)$$

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \quad (11)$$

By comparing eqn 10 & 11

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$G = \mu$$

$$K = \lambda + \frac{2}{3}\mu$$

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Relations between Elastic Constants

$$K = \frac{E}{3(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

$$G = \frac{3K(1-2\nu)}{2(1+\nu)}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Strain Components in terms of stress:

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \\
\varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \\
\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad & \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G}
\end{aligned}\right\} \quad (12)$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \mathrm{x} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

Stress Components from strain components:

Adding the first 3 equations of eq. 12
we get

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = \frac{1-2\nu}{E}\left[\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right] \qquad (13)$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} + \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{xx}}{E} - \nu\frac{\sigma_{yy}}{E} - \nu\frac{\sigma_{zz}}{E}$$

$$\varepsilon_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$

Using eq. 13

$$\varepsilon_{xx} = \frac{1+\nu}{E}\sigma_{xx} - \frac{\nu}{1-2\nu}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$\sigma_{xx} = \frac{E}{(1+\nu)}\varepsilon_{xx} + \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

$$\sigma_{yy} = \frac{E}{(1+\nu)}\varepsilon_{yy} + \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

$$\sigma_{zz} = \frac{E}{(1+\nu)}\varepsilon_{zz} + \frac{E\nu}{(1+\nu)(1-2\nu)}\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right)$$

$$\tau_{xy} = G\gamma_{xy} \qquad \tau_{xz} = G\gamma_{xz} \qquad \tau_{yz} = G\gamma_{yz}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E\nu}{(1+\nu)(1-2\nu)} & \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \text{x} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

HOOKE'S LAW FOR ISOTROPIC MATERIALS

The state of stress at a point is given by σxx = 120 MPa; σyy = 55MPa; σzz = -85MPa;

τxy = -55 MPa; τyz = 33 MPa; τxz = -75MPa. Find the strain components

Take E = 2.07 x 10^5 MPa; γ = 0.3

The strain tensor at a point is given as $\begin{bmatrix} 2 & 7 & 6 \\ 7 & 16 & 0 \\ 6 & 0 & 4 \end{bmatrix}$ x 10-4. Determine the principal stress and the corresponding principal plane.

BOUNDARY CONDITIONS

BOUNDARY CONDITIONS

BOUNDARY CONDITIONS

- Equilibrium Condition
- Strain Displacement Relations
- Constitutive Relations or Generalized Hooke’s Law
- Compatibility Relations

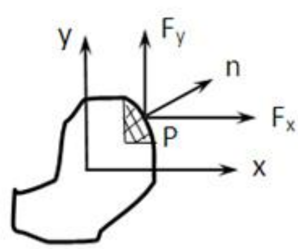
BOUNDARY CONDITIONS

Differential equation of equilibrium must be satisfied throughout the volume of the body.

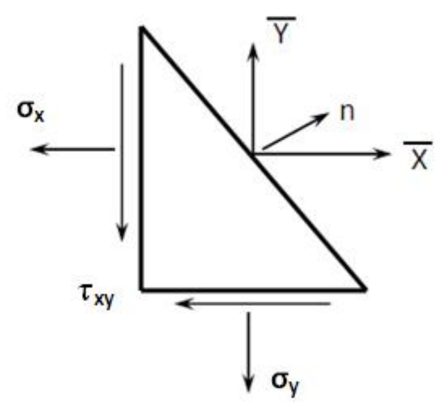
The stress components vary over the volume of the body and when we arrive at the boundary the stress distribution must be such that they should be in equilibrium with external forces on the boundary of the body.

Thus external forces may be considered as a continuation of internal stress distribution.

BOUNDARY CONDITIONS



Element near a Boundary



Free Body Diagram

BOUNDARY CONDITIONS

Since F_x, F_y must be a continuation of σ_x, σ_y & τ_{xy}.

Using Cauchy's equation

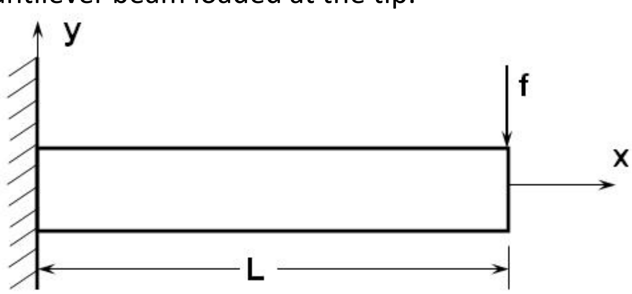
$$\mathbf{T_x^n = \sigma_x n_x + \tau_{xy} n_y = \bar{X}}$$

$$\mathbf{T_y^n = \tau_{yx} n_x + \sigma_y n_y = \bar{y}}$$

Where $\bar{\mathbf{X}}$ and $\bar{\mathbf{y}}$ are surface forces per unit area.

BOUNDARY CONDITIONS

Eg: Cantilever beam loaded at the tip.



At x = 0
u = 0
v = 0
θ_z = 0

At x = L
F_y = -f
F_x = 0

BOUNDARY CONDITIONS

In three dimensional state of stress.

$$T_x^n = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = \bar{X}$$

$$T_y^n = \tau_{yx} n_x + \sigma_y n_y + \tau_{yz} n_z = \bar{y}$$

$$T_z^n = \tau_{zx} n_x + \tau_{zy} n_y + \sigma_z n_z = \bar{z}$$

Where $\bar{z}$ is the external surface force/ unit area in the z direction.

PROBLEMS IN ELASTICITY

- Equilibrium Condition
- Strain Displacement Relations
- Constitutive Relations or Generalized Hooke's Law
- Compatibility Relations
- Boundary Conditions

PROBLEMS IN ELASTICITY

2-D PROBLEMS IN ELASTICITY

PLANE STRESS PROBLEMS IN ELASTICITY

PLANE STRESS PROBLEMS

PLANE STRESS PROBLEMS IN ELASTICITY

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \qquad \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

GENERALIZED STATE OF STRESS PLANE STRESS

PLANE STRESS PROBLEMS IN ELASTICITY

Examples for plane stress problems

- Thin plate loaded by forces at the boundary parallel to the plate.
- Laterally Loaded Beams
- Rotating Discs

PLANE STRESS PROBLEMS IN ELASTICITY

- Equilibrium Equation

1. $\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + B_x = 0$
2. $\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + B_y = 0$

- Strain – Displacement Equations

1. $\varepsilon_{xx} = \frac{\partial u}{\partial x}$
2. $\varepsilon_{yy} = \frac{\partial v}{\partial y}$
3. $\varepsilon_{zz} = \frac{\partial w}{\partial z}$
4. $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$
5. $\gamma_{yz} = 0$
6. $\gamma_{xz} = 0$

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PLANE STRESS PROBLEMS IN ELASTICITY

Hooke's Law

1. $\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right]$
2. $\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$
3. $\varepsilon_{zz} = \frac{-\nu}{E}\left[\sigma_{xx} + \sigma_{yy}\right]$
4. $\gamma_{xy} = \frac{\tau_{xy}}{G}$

Compatibly Equations

1. $\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$
2. $\frac{\partial^2\varepsilon_{zz}}{\partial x^2} = 0 \qquad \frac{\partial^2\varepsilon_{zz}}{\partial y^2} = 0 \qquad \frac{\partial^2\varepsilon_{zz}}{\partial x\partial y} = 0$

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PLANE STRAIN PROBLEMS IN ELASTICITY

PLANE STRAIN PROBLEMS

PLANE STRAIN PROBLEMS IN ELASTICITY

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} \\ \gamma_{yx} & \varepsilon_{yy} \end{bmatrix}$$

GENERALIZED STATE OF STRAIN PLANE STRAIN

PLANE STRAIN PROBLEMS IN ELASTICITY

Examples for plane strain problems

- Retaining wall with a lateral load
- Long Cylinder subjected to internal & external pressure.

PLANE STRAIN PROBLEMS IN ELASTICITY

Equilibrium Equation

1. $$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0$$
2. $$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0$$

Strain – Displacement Equations

1. $$\varepsilon_{xx} = \frac{\partial u}{\partial x}$$
2. $$\varepsilon_{yy} = \frac{\partial v}{\partial y}$$
3. $$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0$$
4. $$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
5. $$\gamma_{yz} = 0$$
6. $$\gamma_{xz} = 0$$

PLANE STRAIN PROBLEMS IN ELASTICITY

Hooke's Law

1. $\varepsilon_{xx} = \frac{1}{2G}\left[\sigma_{xx} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$
2. $\varepsilon_{yy} = \frac{1}{2G}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$
3. $\gamma_{xy} = \frac{1}{2G}\tau_{xy}$

Compatibly Equation

1. $\frac{\partial^2\varepsilon_{xx}}{\partial y^2} + \frac{\partial^2\varepsilon_{yy}}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$

Expression for σ_z:

$$\sigma_{zz} = \nu\left(\sigma_{xx} + \sigma_{yy}\right)$$

SOLUTION OF PLANE PROBLEMS IN ELASTICITY

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SOLUTION OF PLANE PROBLEMS IN ELASTICITY

A plane problem in elasticity can be solved in terms of displacement. The number of unknown is 8

viz. $u, v, \varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \sigma_{xx}, \sigma_{yy}, \tau_{xy}$

The number of equations is also 8

- Strain – Displacement Relations – 3
- Equations of Equilibrium – 2
- Stress – Strain Relations – 3

SOLUTION OF PLANE PROBLEMS IN ELASTICITY

A plane problem in elasticity can also be solved in terms of stress. The number of unknown is 6

viz. $\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \sigma_{xx}, \sigma_{yy}, \tau_{xy}$

The number of equations is also 6

- Equations of Equilibrium – 2
- Stress – Strain Relations – 3
- Compatibility Equations - 1

AIRY’S STRESS FUNCTION

AIRY’S STRESS FUNCTION

AIRY’S STRESS FUNCTION

A plane problem in elasticity can be solved by introducing a new function called Airy’s Stress Function

Airy’s Stress Function (ϕ) can be defined as:

$$\frac{\partial^2 \Phi}{\partial y^2} = \sigma_{xx} - V$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \sigma_{yy} - V$$

Where V(x,y) is a potential. If the body force distribution is assumed as conservative,

AIRY'S STRESS FUNCTION

$$\mathbf{B_x} = -\frac{\partial \mathbf{V}}{\partial \mathbf{x}}$$

$$\mathbf{B_y} = -\frac{\partial \mathbf{V}}{\partial \mathbf{y}}$$

B_x and B_y are body force along x and y directions per unit volume.

AIRY'S STRESS FUNCTION

SOLVING PLANE STRESS PROBLEMS USING AIRY'S STRESS FUNCTION (Φ)

AIRY'S STRESS FUNCTION

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{------ (1)} \quad \text{Compatibility Eqn.}$$

$$\left.\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu\sigma_{yy}] \quad \text{------ (2)} \\
\varepsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu\sigma_{xx}] \quad \text{------ (3)} \\
\gamma_{xy} &= \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \quad \text{------ (4)}
\end{aligned}\right\} \text{Hooke's Law}$$

Substituting for ε_{xx}, ε_{yy} and γ_{xy} from eqns 2,3 & 4 in eqn.1 we get

AIRY'S STRESS FUNCTION

$$\frac{\partial^2}{\partial y^2}(\sigma_{xx} - \nu\sigma_{yy}) + \frac{\partial^2}{\partial x^2}(\sigma_{yy} - \nu\sigma_{xx}) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \quad \text{------ (a)}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \quad \text{------ (5)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0 \quad \text{------ (6)}$$

Differentiating eqn. 5 w.r.t x , eqn. 6 w.r.t y and adding them gives

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \quad \text{------ (b)}$$

AIRY'S STRESS FUNCTION

Substituting $2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$ of eqn. (b) in eqn. (a) gives

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu \cancel{\frac{\partial^2 \sigma_{yy}}{\partial y^2}} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu \cancel{\frac{\partial^2 \sigma_{xx}}{\partial x^2}} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}$$
$$- \nu \cancel{\frac{\partial^2 \sigma_{xx}}{\partial x^2}} - \nu \cancel{\frac{\partial^2 \sigma_{yy}}{\partial y^2}} - \nu \frac{\partial B_x}{\partial x} - \nu \frac{\partial B_y}{\partial y}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \left(\sigma_{xx} + \sigma_{yy}\right) = -(1+\nu)\left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right] \qquad \text{------} \quad (7)$$

AIRY'S STRESS FUNCTION

Expressing σ_{xx}, σ_{yy} in terms of stress function Φ and B_x and B_y in terms of potential V

$$\nabla^4 \phi + 2\nabla^2 V = (1+\nu)\nabla^2 V$$

$$\nabla^4 \phi = -(1-\nu)\nabla^2 V$$

In the absence of body forces the above eqn. becomes

$$\nabla^4 \phi = 0$$

Eqn. 7 is called compatibility eqn. in terms of stress function Φ

AIRY'S STRESS FUNCTION

SOLVING PLANE STRAIN PROBLEMS USING AIRY'S STRESS FUNCTION (Φ)

AIRY'S STRESS FUNCTION

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{------} \quad (1) \quad \text{Compatibility Eqn.}$$

$$\varepsilon_{xx} = \frac{1+\nu}{E}[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})] \quad \text{------} \quad (2)$$

$$\varepsilon_{yy} = \frac{1+\nu}{E}[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})] \quad \text{------} \quad (3)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \quad \text{------} \quad (4)$$

Hooke's Law

Substituting for ε_{xx}, ε_{yy} and γ_{xy} from eqns 2,3 & 4 in eqn.1 we get

AIRY'S STRESS FUNCTION

$$\frac{\partial^2}{\partial y^2}\left[\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy})\right] + \frac{\partial^2}{\partial x^2}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{yy})\right] = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \text{ ---- (a)}$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + B_x = 0 \text{ ---- (5)}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0 \text{ ---- (6)}$$

Differentiating eqn. 5 w.r.t x , eqn. 6 w.r.t y and adding them gives

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y} \text{ ---- (b)}$$

AIRY'S STRESS FUNCTION

Substituting $2\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$ of eqn. (b) in eqn. (a) gives

$$\frac{\partial^2 \sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2(\sigma_{xx}+\sigma_{yy})}{\partial y^2} + \frac{\partial^2 \sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2(\sigma_{xx}+\sigma_{yy})}{\partial x^2} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) - \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = \frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_{xx} + \sigma_{yy}) = -\frac{1}{1-\nu}\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \text{ ---- (7)}$$

AIRY'S STRESS FUNCTION

Expressing σ_{xx}, σ_{yy} in terms of stress function Φ and B_x and B_y in terms of potential V

$$\bar{\nabla}^2\left(\frac{\partial^2\Phi}{\partial y^2}+V+\frac{\partial^2\Phi}{\partial x^2}+V\right)=-\frac{1}{1-\nu}\left(-\frac{\partial^2 V}{\partial x^2}-\frac{\partial^2 V}{\partial y^2}\right)$$

$$\bar{\nabla}^2\bar{\nabla}^2\Phi=\left(\frac{1}{1-\nu}-2\right)\bar{\nabla}^2 V$$

$$\bar{\nabla}^4\Phi=\frac{-1+2\nu}{1-\nu}\bar{\nabla}^2 V$$

In the absence of body forces the above eqn. becomes

$$\bar{\nabla}^4\Phi=0$$

Eqn. 7 is called compatibility eqn. in terms of stress

SOLUTION BY POLYNOMIAL

SOLUTION BY POLYNOMIAL

SOLUTION BY POLYNOMIAL

When the body forces are absent, the solution of a 2-D problem in elasticity will get reduced to integrating the differential equation:

$$\frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial y^2} + \frac{\partial^4\Phi}{\partial y^4} = 0$$

Solutions in the form of polynomials are suitable for long rectangular stripes. Suitably adjusting the co-efficents considering the boundary condition a number of practically important problems can be solved.

SOLUTION BY POLYNOMIAL

Polynomial of Second Degree:-

$$\Phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2$$

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2} = c_2$$

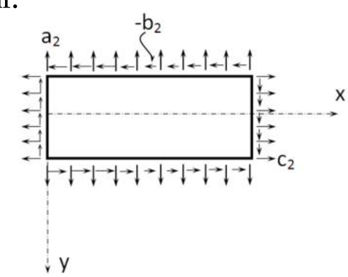
$$\sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2} = a_2$$

$$\tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} = -b_2$$

Represents a state of uniform tension or compression in two perpendicular directions and a uniform shear.

SOLUTION BY POLYNOMIAL

The stress components are constant throughout the body. Thus stress function Φ_2 represents a combination of uniform tension or compression in two perpendicular direction and a uniform shear.



A Rectangular Plate subjected uniform tension in two perpendicular directions and a uniform shear.

SOLUTION BY POLYNOMIAL

Polynomial of Third Degree:-

$$\Phi_3 = \frac{a_3}{3(2)}x^3 + \frac{b_3}{2}x^2y + \frac{c_3}{2}xy^2 + \frac{d_3}{3(2)}y^3$$

The above stress function Φ_3 satisfies eqn. I

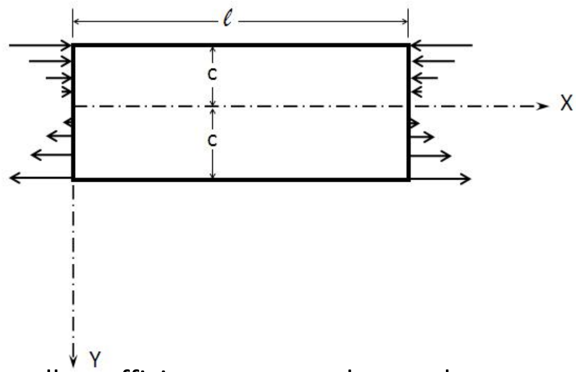
$$\sigma_{xx} = \frac{\partial^2\Phi_3}{\partial y^2} = c_3x + d_3y$$

$$\sigma_{yy} = \frac{\partial^2\Phi_3}{\partial x^2} = a_3x + b_3y$$

$$\tau_{xy} = -\frac{\partial^2\Phi_3}{\partial x\partial y} = -b_3x - c_3y$$

SOLUTION BY POLYNOMIAL

Polynomial of Third Degree:-



Assuming all coefficients except d_3 equal to zero, we obtain pure bending.

Assuming all coefficients except a_3 equal to zero, we obtain pure bending by normal stress applied to the sides y = ±C of the plate

SOLUTION BY POLYNOMIAL

Polynomial of Third Degree:-

Assuming all coefficients except b_3 or c_3 equal to zero, we obtain normal and shear stress acting on the sides of the plate.

$\sigma_{yy} = b_3y$

$\sigma_{xx} = 0$

$\tau_{xy} = -b_3x$

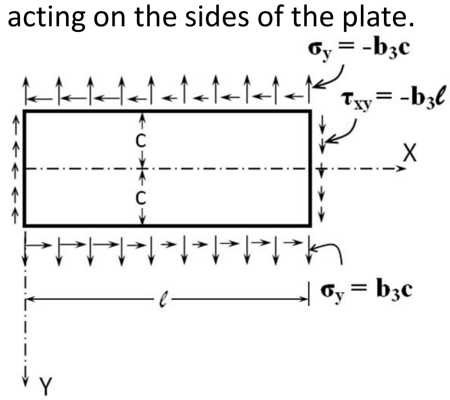


Figure represents the case in which b_3 alone is set not equal to zero

SOLUTION BY POLYNOMIAL

Polynomial of Fourth Degree:-

$$\Phi_4 = \frac{a_4}{4(3)}x^4 + \frac{b_4}{3(2)}x^3y + \frac{C_4}{2}x^2y^2 + \frac{d_4}{3(2)}xy^3 + \frac{e_4}{4(3)}y^4$$

$$e_4 = -(2C_4 + a_4)$$

$$\sigma_{xx} = \frac{\partial^2\Phi_4}{\partial y^2} = c_4x^2 + d_4xy - (2C_4 + a_4)y^2$$

$$\sigma_{yy} = \frac{\partial^2\Phi_4}{\partial x^2} = a_4x^2 + b_4xy + C_4y^2$$

$$\tau_{xy} = -\frac{\partial^2\Phi_3}{\partial x\partial y} = -\frac{b_4}{2}x^2 - 2c_4xy - \frac{d_4}{2}y^2$$

SOLUTION BY POLYNOMIAL

Polynomials of second, third and fourth degree which satisfies the governing differential eqn. $\nabla^4\Phi = 0$ have been discussed above. Various boundary conditions are obtained by conveniently choosing coefficients. Since the governing differential equation is a linear differential eqn. sum of several solutions is also a solution. We can superimpose elementary solutions so as to satisfy boundary condition. This can be used to solve 2D problem in elasticity.

SOLUTION BY POLYNOMIAL

Once the stress function Φ has been obtained, σ_{xx}, σ_{yy}, τ_{xy} are determined by taking the suitable derivatives, strain components ε_{xx}, ε_{yy} and $\gamma_{xy.}$ can be obtained using Hooke's law. The displacement components u, v and w are obtained by using the strain displacement relations.

Bending of a Cantilever with an End Load:

The beam is considered as thin and it can be analysed using plane stress concept. At any section along the length of the cantilever the bending stress σ_{xx} varies linearly with y. This stress pattern can be obtained by taking the term d_4xy^3 from the stress function of degree 4.

The transverse load produces a shear stress distribution varying in the y direction with a non zero shear stress at y = 0. The term d_4xy^3 give rise to a shear stress distribution with zero shear stress at y = 0. To introduce a non zero shear stress at y =0, the term b_2xy from second degree polynomial is added.

Thus we choose the function given below to solve the bending of a cantilever with end load.

$$\emptyset(x, y) = b_2xy + d_4xy^3$$

The above equation satisfies the bi-harmonic equation

Also, $\sigma_{yy} = \frac{\partial^2\emptyset}{\partial x^2} = 0$

$$\sigma_{xx} = \frac{\partial^2\emptyset}{\partial y^2} = \frac{\partial^2(b_2xy + d_4xy^3)}{\partial y^2} = 6d_4xy$$

$$\tau_{xy} = \frac{\partial^2\emptyset}{\partial x\,\partial y} = \frac{\partial\left(\frac{\partial\emptyset}{\partial x}\right)}{\partial y} = \frac{\partial(b_2x + d_4y^3)}{\partial y} = -(b_2 + 3d_4y^2)$$

Applying the boundary conditions:

Stress free bottom and top layers i.e.,

$$\tau_{xy} = 0 \quad @ \quad y = \pm h$$

$$-(b_2 + 3d_4h^2) = 0$$

$$d_4 = \frac{-b_2}{3h^2}$$

$$\tau_{xy} = \frac{b_2}{h^2}(y^2 - h^2)$$

Sum of total shear force on any section is equal to the applied load P.

$$\int_{-h}^{h} \tau_{xy} t\,dy = P$$

Substitution and simplification gives:

$$\frac{tb_2}{h^2}\int_{-h}^{h}(y^2 - h^2)\,dy = P$$

$$b_2 = -\frac{3P}{4th}$$

$$d_4 = \frac{-b_2}{3h^2} = \frac{P}{4th^3}$$

$$\emptyset(x,y) = \frac{P}{4th}\left[\frac{xy^3}{h^2} - 3xy\right]$$

$$\sigma_{xx} = \frac{3P}{2th^3}xy$$

$$\tau_{xy} = \frac{3P}{4th^3}(h^2 - y^2)$$

$$\sigma_{xx} = \frac{P}{I}xy$$

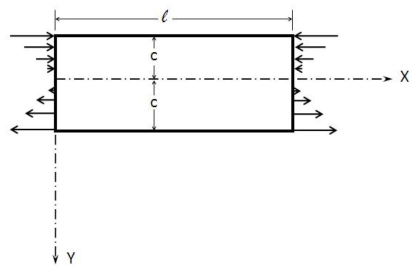
$$\tau_{xy} = \frac{P}{I}\frac{1}{2}(h^2 - y^2)$$

SAINT VENANT'S PRINCIPLE

SAINT VENANT'S PRINCIPLE

SAINT VENANT'S PRINCIPLE

In the case of solution by polynomials, solution is obtained from very simple forms of stress function. In this case the boundary forces must be distributed exactly as solution itself requires. In the case of pure bending the load on the ends are applied as shown in figure below.



SAINT VENANT'S PRINCIPLE

If the couples on the ends are applied in any other manner, a new solution must be found if the changed boundary conditions on the ends are to exactly satisfied. Many such solutions have been found out and it has been seen that a change in the distribution of load on an end without change of the resultant alters the stress significantly near the end.

Saint Venant principle states that except in the immediate vicinity of the point of application of loads, the stress distribution may be assumed independent of actual mode of application of load

Assignment Questions

Solve the following problems using the solution by polynomials

a. Bending of cantilever beam loaded at the end
b. Bending of a beam by uniform load